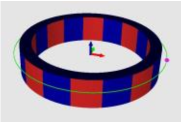

E. Computation of Permanent Magnetic Fields

The following passages should give an impression, how permanent magnets can be calculated in respect of their field distribution. This overview certainly cannot cover all subjects. It will merely introduce the basic differential equations of magnetostatics, provide a quick glance onto the numerical method of Finite Elements and will then explain the most popular methods to get fields of magnets analytically. Some formulas will be presented, which are based on the theory of vector analysis. So the meaning of some symbols which are in use will be depicted in an appendix at the end of this chapter.

1. General

The root of all formulas for the analysis of macroscopic magnetic systems are the Maxwell equations together with some material laws. From these Maxwell equations partial differential equations of electromagnetic potentials can be derived. Those cover all fields of EM phenomena like static and time dependent electric and magnetic fields, current distributions, electric circuits or wave phenomena. Here first an equation for static permanent magnets together with DC currents will be derived.

Eq. (B.4) was:

$$\vec{B} = \vec{B}_r + \mu_0 \cdot \mu_r(H) \cdot \vec{H} \qquad \text{(E.1)}$$

This constitutive relation includes permanent magnets with remanence induction $\mathbf{B_r}$ and a general field dependent but relatively small permeability μ_r. But it also can stand for soft magnetic materials, where $\mathbf{B_r}$ is relatively small and μ_r can be very large.

We now use the vector potential **A** and in addition $\mathbf{M_r}$ instead of $\mathbf{B_r}$, i.e.:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{M}_r = \vec{B}_r / \mu_0 \qquad \text{(E.2)}$$

The use of **A** follows from $\nabla \cdot \mathbf{B}=0$, compare (EA.8) in the appendix. Applying the rotation operator at both sides of (E.1) and using (E.2) provides:

$$\vec{\nabla} \times \frac{1}{\mu_r} \vec{\nabla} \times \vec{A} = \mu_0 \cdot (\vec{\nabla} \times \vec{H} + \vec{\nabla} \times \frac{\vec{M}_r}{\mu_r}) \qquad \text{(E.3)}$$

Remembering eq. (A.4) we see, that the first term at the right side is nothing else than the current density **j**.

$$\vec{\nabla} \times \frac{1}{\mu_r} \vec{\nabla} \times \vec{A} = \mu_0 \cdot (\vec{j} + \vec{\nabla} \times \frac{\vec{M}_r}{\mu_r}) \qquad \text{(E.4)}$$

Known parameters for this magnetostatic formulation are the DC current density, the distribution of remanent magnetization as well as permeabilities of hard or soft magnetic materials. The vector potential **A** is the unknown variable. In addition to partial differential equations as (E.4), additional constraints for the vector potential have to be demanded. These are relations for its behavior at the spatial borders of the problem, as well as gauge relations, see below.

They consider that **A** is not defined uniquely by only the differential equation. Solutions of (E.4) for **A** and so for **B**, **H** etc. are analytically available for only a few special cases, which means that numerical methods have to be applied then. The most popular one is nowadays the Finite Element Method (FEM).

The Finite Element Method separates space into small elements, assuming a linear or polynomial behavior of the components of **A** in each element. Fig. E1 shows such an element distribution for a two dimensional example. The behavior of the potential in an element is parametrized with the help of its values on the nodes or edges of the element. Under these conditions (E.4) can be reformulated to a system of linear equations with the node or edge values of **A** as unknowns. The Finite Element Method so provides an approximation of reality which increases in quality with growing number of elements. Disadvantageous in this method is, that FEM software packages are very expensive on one hand, which means several 10.000 US$ for 3D packages. In addition most of them demand a high grade of training and the analyses are fairly time consuming in most cases.

Beside the direct solution of differentials equations and beside the FEM method, several other numerical methods like FDM (Finite Difference Method), BEM (Boundary Element Method) or FIT (Finite Integration Technique) exist, but are of less popularity or have to suffer from a few disadvantages compared to FEM.

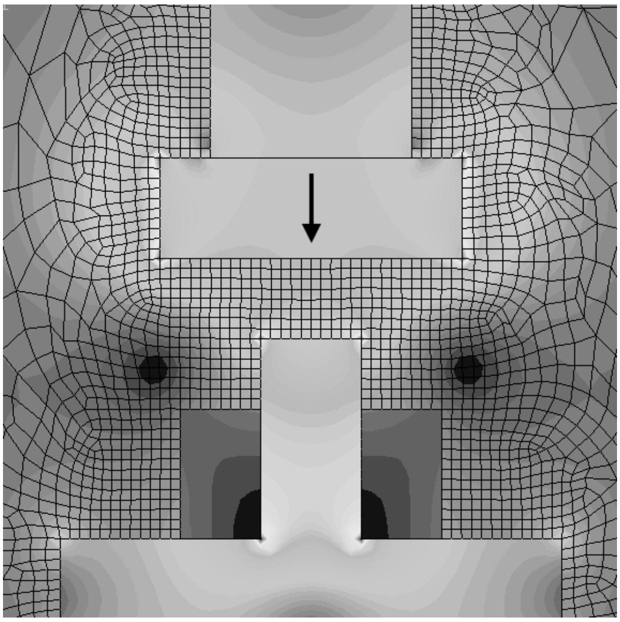

Fig. E1: Current coil and unidirectional permanent magnet surrounded by iron and air, analyzed with FEM method.

As another approximation method the so called method of magnetic circuits is mentioned in nearly every book about magnetism. This was in extensive use in the past before FEM together with cheap computer resources became available. This method is mostly used to treat nearly closed magnetic circuit systems, that consist of magnets and current conductors which are embedded into soft magnetic materials To get good results here the existence of only small air gaps has to be demanded. Please refer to literature for further information.

After this general introduction, we will focus on the mathematical treatment of three dimensional systems that consist only of permanent magnets.

2. Treatment of Mere Permanent Magnets

In absence of DC currents and soft magnetic materials in (E.4) there is **j**=0. μ_r is close to one for the most cases of hard magnetic materials. So (E.4) reduces to the following formula:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{\nabla} \times \vec{M}_r$$

Now we introduce a gauge equation for the vector potential, here the so called Coulomb gauge:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{E.5}$$

Together with the vector identity (EA.9) from the appendix below, the above then becomes

$$\Delta\vec{A} = -\mu_0 \cdot \vec{\nabla} \times \vec{M}_r \tag{E.6}$$

This differential equation has a general solution by eq. (E.7):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{\nabla}' \times \vec{M}_r}{\left|\vec{r} - \vec{r}'\right|} dV' + \frac{\mu_0}{4\pi} \oint_F \frac{\vec{M}_r \times \vec{n}'}{\left|\vec{r} - \vec{r}'\right|} dF' \tag{E.7}$$

Here **r** is the location were the field has to be calculated and **r’** is the vector of the magnets locations. The integrations are done over the magnets volume V as well as over the magnets surface F.

Eq. (E.7) is of less popularity than the following formulation, where instead of the vector potential a scalar potential is used.

Taking eq. (E.1) together with setting μ_r=1 for the magnet leads together with (E.2) to:

$$\vec{B} = \mu_0 \cdot M_r + \mu_0 \cdot \vec{H}$$

Taking into account that **B** has no sources, i.e. **∇**·**B**=0 see eq. (A.3), this leads to

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}_r \tag{E.8}$$

Since **j**=0. i.e. **∇**×**H**=0, vector analysis provides that **H** can be expressed by a scalar potential, see eq. (EA.7) of the appendix:

$$\vec{H} = -\vec{\nabla}\Phi \tag{E.9}$$

This provides together with (E.8) in the absence of currents:

$$\Delta\Phi = \vec{\nabla} \cdot \vec{M}_r \tag{E.10}$$

The general solution of (E.10) is given by:

$$\Phi(\vec{r}) = -\frac{1}{4\pi} \int_V \frac{\vec{\nabla}' \cdot \vec{M}_r}{\left|\vec{r} - \vec{r}'\right|} dV' + \frac{1}{4\pi} \oint_F \frac{\vec{M}_r \cdot \vec{n}'}{\left|\vec{r} - \vec{r}'\right|} dF' \tag{E.11}$$

The integration is performed again over the magnets volume as well as over its surfaces.

Which of both formulations i.e. the vector or scalar potential formulation is taken, often depends on the ease of solving the respective integrals and may be different for different geometries. In general, often the integrals can not be expressed by explicit formulas but have to be treated numerically.

3. Example for the Use of the Scalar Potential Formulation

In the following we will show an easy example of the application of a magnetic scalar potential on a homogeneously magnetized cylinder or disc magnet with axial height h and radius r_a. The field shall be computed at a distance d from the magnets surface, see the sketch below. The magnetization is oriented in axial direction, i.e. $\mathbf{M}=M_r{*}\mathbf{e_z}$.

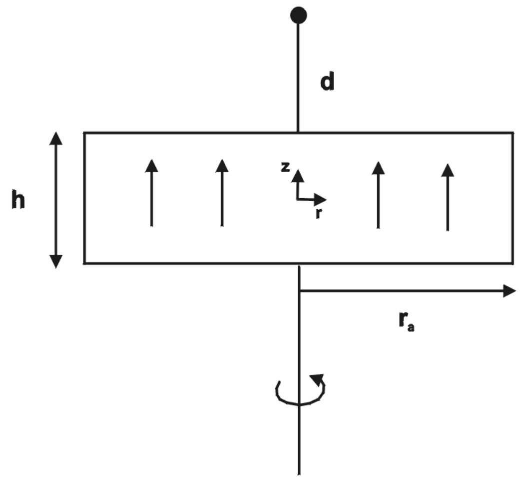

Fig.E2: Sketch of a homogeneously magnetized flat cylinder. The field shall be computed at a point with distance d from its upper face.

When we take the potential solution eq. (E.11) we see that inside the volume integral there is $\nabla\cdot\mathbf{M}_r=0$, so that the volume integral itself vanishes. This is always the case with a homogeneous magnetization, so the fields are originated only by the pole faces.

For simplification we combine the rest of eq. (E.11) with eq. (E.9) where the Nabla operator can be taken under the integral. So we get for the field:

$$H(\vec{r}) = -\frac{1}{4\pi}\oint_F \vec{M}_r\cdot\vec{n}'\,\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dF' \qquad \text{(E.12)}$$

The scalar product of magnetization with the surface's normal vector is only non zero at the head faces of the cylinder, were the plus sign is valid at the top and minus at the bottom side:

$$\vec{M}_r\cdot\vec{n} = \pm M_r \qquad \text{(E.13)}$$

The surface element of a head face is

$$dF' = r'dr'd\varphi' \qquad \text{(E.14)}$$

From symmetry it follows that at the center point at distance d, there can only be an axial, i.e. z-component of the field. So we only need the z-part of the Nabla operator under the integral:

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} \rightarrow \frac{\partial}{\partial z}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\partial}{\partial z}\frac{1}{\sqrt{r'^2+(z-z')^2}}$$

z'=+/- h/2 is the location of the head faces, and z is the coordinate of the point of interest. Both z' and z are related to the coordinate origin, which is located at the center of the magnet. After differentiation and expressing both z' and z with the help of h and d one gets:

$$\frac{\partial}{\partial z}\frac{1}{\left|\vec{r}-\vec{r}'\right|}=-\frac{d}{\sqrt{(r'^2+d^2)^3}} \quad \text{for the upper face} \tag{E.15a}$$

and

$$\frac{\partial}{\partial z}\frac{1}{\left|\vec{r}-\vec{r}'\right|}=-\frac{(d+h)}{\sqrt{(r'^2+(d+h)^2)^3}} \quad \text{for the lower face} \tag{E.15b}$$

Now we summarize (E.12)-(E.15) and get for the field:

$$\vec{H}(d) = H_z \cdot \vec{e}_z$$

with (E.16)

$$H_z = H_{zt} - H_{zb}$$

where

$$H_{zt} = \frac{M_r}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{ra} \frac{d \cdot r'}{\sqrt{(r'^2+d^2)^3}} dr' \tag{E.16a}$$

and

$$H_{zb} = \frac{M_r}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{ra} \frac{(d+h) \cdot r'}{\sqrt{(r'^2+(d+h)^2)^3}} dr' \tag{E.16b}$$

The integration over the angle is elementary and the integrals over the radius can be taken from integral maps. Doing this the final result can be rewritten to

$$\vec{H}(d) = H_z \cdot \vec{e}_z \tag{E.17}$$

with

$$H_z(d) = \frac{M_r}{2}(g(d+h) - g(d)) \tag{E.17a}$$

where

$$g(w) = \frac{w}{\sqrt{r_a{}^2 + w^2}} \qquad \text{E.17b)}$$

E.g. a magnet with ra=5mm, h=3mm and Mr=800kA/m (=1T) originates a field of 171 kA/m at a distance of 1mm.

Whereas this example shows a result which can be obtained quite easily, in general the integrals can not be solved by explicit expressions and have to be treated numerically. In the case of non homogeneous distributions of magnetization additionally the volume integrals of eq. (E.11) have to be solved, which demand additional efforts.

Appendix

In the above some symbols of vector analysis like the Nabla operator were used and shall be depicted here.

The Nabla operator can be introduced in Cartesian coordinates as a vector of single component differential operators:

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{EA.1}$$

In Cartesian coordinates this can be applied like a vector in the form of a dot product and cross product with other vectors. With scalar fields it can by applied by simple multiplication. So entities like divergence, rotation and gradient can be formed:

$$\vec{\nabla} \cdot \vec{a} = \mathrm{div}(\vec{a}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \tag{EA.2}$$

$$\vec{\nabla} \times \vec{a} = \mathrm{rot}(\vec{a}) = \vec{e}_x (\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}) + \vec{e}_y (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}) + \vec{e}_z (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}) \tag{EA.3}$$

$$\vec{\nabla}\Phi = \mathrm{grad}\Phi = \vec{e}_x \frac{\partial}{\partial x}\Phi + \vec{e}_y \frac{\partial}{\partial y}\Phi + \vec{e}_z \frac{\partial}{\partial z}\Phi \tag{EA.4}$$

Since we used grad(Φ) in cylinder coordinates r,φ,z in the example above, we will give it here also in this coordinate system. Expressions for div(**a**) and rot(**a**) in other coordinate systems can be found in literature.

$$\vec{\nabla}\Phi = \mathrm{grad}\Phi = \vec{e}_r \frac{\partial}{\partial r}\Phi + \frac{1}{r}\vec{e}_\varphi \frac{\partial}{\partial \varphi}\Phi + \vec{e}_z \frac{\partial}{\partial z}\Phi \tag{EA.5}$$

Another operator following from Nabla is the Laplace operator. In Cartesian coordinates:

$$\Delta = \vec{\nabla}^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} \tag{EA.6}$$

Some important characteristics of vector fields in relation with the above operators are as follows:

$$\vec{\nabla} \times \vec{a} = 0 \Rightarrow \vec{a} = \vec{\nabla}\Phi \tag{EA.7}$$

(In words: When a vector field is curl free it can be expressed as the gradient of a scalar potential)

$$\vec{\nabla} \cdot \vec{a} = 0 \Rightarrow \vec{a} = \vec{\nabla} \times \vec{A} \tag{EA.8}$$

(In words: When a vector field is source free it can be expressed as the curl (rotation) of a vector potential)

One identity between different operator expressions of a vector field which was used above is the following:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{a} = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \Delta\vec{a} \tag{EA.9}$$

A lot of other identities between the above operators in reference to scalar and vector fields can be found in literature. Also refer to literature with respect to the evaluation of surface and volume integrals as well as to relations between them, like the Stokes, Gauss or Greens relations.