E. Computation of Permanent Magnetic Fields

The following passages should give an impression, how permanent magnets can be analyzed in regard to their field distribution. This overview can not cover all areas. It will merely introduce the basic differential equations of magnetostatics, give a short view to the numerical method of Finite Elements and will then explain the most popular methods to get the fields of magnets analytically. Some formulas will be presented which are based mathematically on the field of vector analysis. So the meaning of some of the symbols being in use here will be depicted in a small appendix at the end of this chapter.

1. General

The heart of all formulas for the analysis of macroscopic magnetic systems are the Maxwell equations together with some material laws. From these Maxwell equations partial differential equations of electromagnetic potentials can be derived, covering all fields of EM phenomena like static and time dependent electric and magnetic fields, current distributions, electric circuits or wave phenomena. At this place first an equation for static permanent magnets together with DC currents will be derived.

Eq.(B.4) was

$$\vec{B} = \vec{B}_r + \mu_0 \cdot \mu_r(H) \cdot \vec{H} \qquad (E.1)$$

This constitutive relation includes permanent magnets with remanence induction **Br** and a general field dependent but relatively small permeability μ_r. But it also can stand for soft magnetic materials, where **Br** is relatively small and μ_r can be very high.

We now use the vector potential **A** and in addition $\mathbf{M_r}$ instead of **Br**, i.e.:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{M}_r = \vec{B}_r/\mu_0 \qquad (E.2)$$

The use of **A** follows from **∇·B**=0, compare (EA.8) in the appendix. Taking the rotation on both sides of (E.1) and using (E.2) supplies:

$$\vec{\nabla} \times \frac{1}{\mu_r} \vec{\nabla} \times \vec{A} = \mu_0 \cdot (\vec{\nabla} \times \vec{H} + \vec{\nabla} \times \frac{\vec{M}_r}{\mu_r}) \qquad (E.3)$$

Remembering eq. (A.4) we see, that the first term on the right side is nothing else than the current density **j**.

$$\vec{\nabla} \times \frac{1}{\mu_r} \vec{\nabla} \times \vec{A} = \mu_0 \cdot (\vec{j} + \vec{\nabla} \times \frac{\vec{M}_r}{\mu_r}) \qquad (E.4)$$

Known parameters for this so called magnetostatic formulation are the DC current density, the distribution of remanence magnetization as well as the permeabilities of hard or soft magnetic materials. **A** is the unknown. In addition to such partial differential equations like (E.4), additional constraints for the vector potential have to be demanded. These are relations for its behavior on the spatial borders of the problem, as well as gauge relations, see below. They consider that **A** is not defined uniquely by only the differential equation. Solutions of (E.4) for **A** and so for **B**, **H** etc. are analytically available for only a few special cases, which means that numerical methods have to be applied. The most popular one is nowadays the Finite Element Method (FEM).

The Finite Element Method separates space into small elements, assuming a linear or quadratic behavior of the components of **A** in each element. Fig. E1 shows such an element distribution for a two dimensional example. The behavior of the potential in an element is parametrized with the help of its values on the nodes or edges of the element. Under these conditions (E.4) can be reformulated to a system of linear equations with the node or edge values of **A** as unknowns. The Finite Element Method so delivers an approximation of reality which increases in quality with growing number of elements. Disadvantageous in this method is, that FEM software packages are very expensive on one hand, which means several 10.000 US$ for 3D packages. In addition most of them demand a high grade of knowledge and the analyses are highly time consuming in most cases.

Beside the direct solution of the differentials equations and the FEM method, several other numerical methods like FDM (Finite Difference Method), BEM (Boundary Element Method) or FIT (Finite Integration Technique) exist, but are of less popularity or have to suffer from big disadvantages compared to FEM.

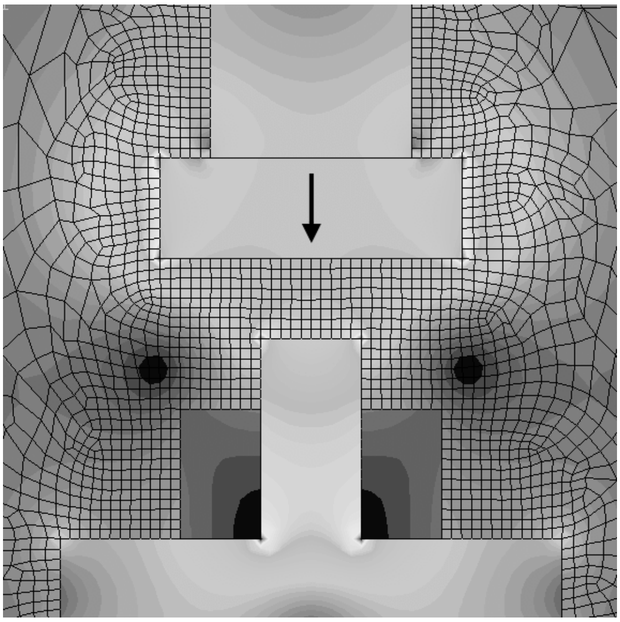

Fig. E1: Interchange of a current coil with an unidirectional permanent magnet (arrow), analyzed by FEM method.

As another approximation method the so called method of magnetic circuits is mentioned in nearly every book about magnetism, which was in extensive use in the past before FEM together with cheap computer resources became available. This method can treat more or less accurately nearly closed systems only, where there are magnets, coils etc. which are embedded in systems of soft magnetic materials. To get good results here the existence of only small air gaps has to be demanded. Please refer to literature for further information.

After this general introduction, we like to approach to the analysis of permanent magnets them self. In the following so we will focus on the mathematical treating of pure permanent magnetic systems in three dimensions

2. Treatment of Permanent Magnets

In absence of DC currents and soft magnetic materials in (E.4) there is **j**=0. μ_r is close to one for the most cases of hard magnetic materials. So (E.4) reduces to the following formula:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{\nabla} \times \vec{M}_r$$

Now we introduce a gauge equation for the vector potential, here the so called Coulomb gauge:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{E.5}$$

Together with the vector identity (EA.9) from the appendix the above then becomes

$$\Delta \vec{A} = -\mu_0 \cdot \vec{\nabla} \times \vec{M}_r \tag{E.6}$$

This set of differential equations has a general solution by eq. (E.7):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{\nabla}' \times \vec{M}_r}{\left|\vec{r} - \vec{r}'\right|} dV' + \frac{\mu_0}{4\pi} \oint_F \frac{\vec{M}_r \times \vec{n}'}{\left|\vec{r} - \vec{r}'\right|} dF' \tag{E.7}$$

Here **r** is the location were the field is sought and **r'** is the vector of the magnets locations. The integrations are done over the magnets volume V as well as over the magnets surface F.

Eq. (E.7) is of less popularity than the following formulation, perhaps for people are scared of the vector potential. This following formulation is that of the magnetic scalar potential:

Taking eq. (E.1) together with setting μ_r=1 for the magnet leads together with (E.2) to:

$$\vec{B} = \mu_0 \cdot M_r + \mu_0 \cdot \vec{H}$$

Taking into account that **B** has no sources, i.e. **∇**·**B**=0 see eq. (A.3), this leads to

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}_r \tag{E.8}$$

Since **j**=0. i.e. **∇**×**H**=0, vector analysis reveals that **H** can be expressed by a scalar potential, see eq. (EA.7) of the appendix:

$$\vec{H} = -\vec{\nabla}\Phi \tag{E.9}$$

This leads together with (E.8) in the absence of currents to:

$$\Delta\Phi = \vec{\nabla} \cdot \vec{M}_r \tag{E.10}$$

The general solution of (E.10) is given by:

$$\Phi(\vec{r}) = -\frac{1}{4\pi} \int_V \frac{\vec{\nabla}' \cdot \vec{M}_r}{\left|\vec{r} - \vec{r}'\right|} dV' + \frac{1}{4\pi} \oint_F \frac{\vec{M}_r \cdot \vec{n}'}{\left|\vec{r} - \vec{r}'\right|} dF' \tag{E.11}$$

The integration is done again over the magnet volume as well as over its surfaces.

Which of both formulations i.e. the vector or scalar potential formulation is taken, often depends on the ease of solving the respective integrals and may be different for different geometries. In general, often the integrals can not be expressed by explicit formulas but have to be treated numerically.

3. Example for the Use of the Scalar Potential Formulation

In the following we will show an easy example of the application of the magnetic scalar potential on a homogeneously magnetized cylinder magnet with axial height h and radius r_a. The field shall be computed at a distance d from the magnets surface, see the sketch below. The magnetization is oriented in axial direction, i.e. $\mathbf{M}=M_r*\mathbf{e_z}$.

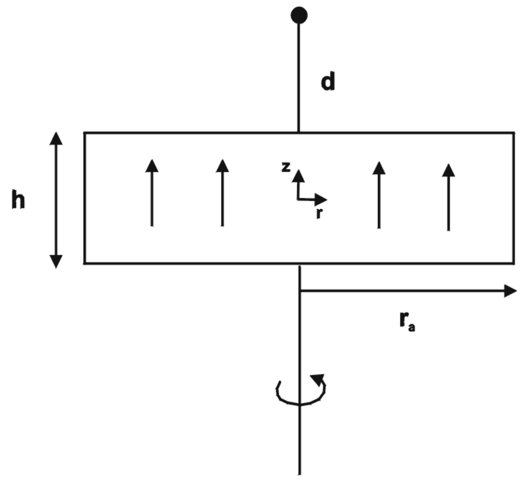

Fig.E2: Sketch of a homogeneously magnetized cylinder. The field shall be computed at a point with distance d from its surface.

When we take the potential solution eq. (E.11) we see that in the volume integral $\nabla\cdot\mathbf{M}_r=0$, so the volume integral itself vanishes. This is always the case with homogeneous kinds of magnetization, so the fields are originated only by the pole surfaces.

For simplification we combine the rest of eq. (E.11) with eq. (E.9) where the Nabla operator can be taken under the integral. So we get for the field:

$$H(\vec{r}) = -\frac{1}{4\pi}\oint_F \vec{M}_r \cdot \vec{n}' \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) dF' \qquad (E.12)$$

The scalar product of magnetization with the surface normal vector is only non zero at the head surfaces of the cylinder, were the plus sign acts at the upper and minus at the lower surface:

$$\vec{M}_r \cdot \vec{n} = \pm M_r \qquad (E.13)$$

The surface element of a head surface is

$$dF' = r'dr'd\varphi' \qquad (E.14)$$

From symmetry it follows that at the center point at distance d, there can only be an axial, i.e. z-component of the field. So we only need the z-part of the Nabla operator under the integral:

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} \rightarrow \frac{\partial}{\partial z}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\partial}{\partial z}\frac{1}{\sqrt{r'^2+(z-z')^2}}$$

z’=+/- h/2 is the location of the head surfaces, and z the coordinate of the point of interest, both related to the coordinate origin, which is located at the center of the magnet. After differentiation and expressing dashed and non dashed z by the help of h and d one gets:

$$\frac{\partial}{\partial z}\frac{1}{\left|\vec{r}-\vec{r}'\right|}=-\frac{d}{\sqrt{(r'^2+d^2)^3}} \quad \text{for upper surface} \tag{E.15a}$$

$$\frac{\partial}{\partial z}\frac{1}{\left|\vec{r}-\vec{r}'\right|}=-\frac{(d+h)}{\sqrt{(r'^2+(d+h)^2)^3}} \quad \text{for lower surface} \tag{E.15b}$$

Now we summarize (E.12)-(E.15) and get for the field:

$$\vec{H}(d) = H_z \cdot \vec{e}_z$$

with (E.16)

$$H_z = H_{zt} - H_{zb}$$

where

$$H_{zt} = \frac{M_r}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{ra} \frac{d \cdot r'}{\sqrt{(r'^2+d^2)^3}} dr' \tag{E.16a}$$

and

$$H_{zb} = \frac{M_r}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{ra} \frac{(d+h) \cdot r'}{\sqrt{(r'^2+(d+h)^2)^3}} dr' \tag{E.16b}$$

The integration over the angle is elementary and the integrals over the radius can be taken from integral maps. Doing this the final result can be rewritten to

$$\vec{H}(d) = H_z \cdot \vec{e}_z \tag{E.17}$$

with

$$H_z(d) = \frac{M_r}{2}(g(d+h) - g(d)) \tag{E.17a}$$

where

$$g(w) = \frac{w}{\sqrt{r_a{}^2 + w^2}} \tag{E.17b}$$

E.g. a magnet with ra=5mm, h=3mm and Mr=800kA/m (=1T) originates a field of 171 kA/m at a distance of 1mm.

Whereas this example shows a result which can be obtained quite easily, in general the integrals can not be solved by explicit expressions and have to be treated numerically. In the case of non homogeneous distributions of magnetization additionally the volume integrals of eq. (E.11) have to be solved, which demand additional efforts.

Appendix

In the above some symbols of vector analysis like the Nabla operator were used and shall be depicted here.

The Nabla operator can be introduced in cartesian coordinates as a vector of single component differential operators:

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{EA.1}$$

In cartesian coordinates this can be applied like a vector in the form of dot product and cross product to vectors. To scalar fields it can by applied by simple multiplication. So entities like divergence, rotation and gradient can be formed:

$$\vec{\nabla} \cdot \vec{a} = \mathrm{div}(\vec{a}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \tag{EA.2}$$

$$\vec{\nabla} \times \vec{a} = \mathrm{rot}(\vec{a}) = \vec{e}_x(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}) + \vec{e}_y(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}) + \vec{e}_x(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}) \tag{EA.3}$$

$$\vec{\nabla}\Phi = \mathrm{grad}\Phi = \vec{e}_x \frac{\partial}{\partial x}\Phi + \vec{e}_y \frac{\partial}{\partial y}\Phi + \vec{e}_x \frac{\partial}{\partial z}\Phi \tag{EA.4}$$

Since we used grad(Φ) in cylinder coordinates r,φ,z in the example above, we will give it here also in this coordinate system. Expressions for div(**a**) and rot(**a**) in other coordinate systems can be found in literature.

$$\vec{\nabla}\Phi = \mathrm{grad}(\Phi) = \vec{e}_r \frac{\partial}{\partial r}\Phi + \frac{1}{r}\vec{e}_\varphi \frac{\partial}{\partial \varphi}\Phi + \vec{e}_z \frac{\partial}{\partial z}\Phi \tag{EA.5}$$

Another operator following from Nabla is the Laplace operator. In cartesian coordinates:

$$\Delta = \vec{\nabla}^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} \tag{EA.6}$$

Some important characteristics of vector fields in relation with the above operators are as follows:

$$\vec{\nabla} \times \vec{a} = 0 \Rightarrow \vec{a} = \vec{\nabla}\Phi \tag{EA.7}$$

(In words: When a vector field is curl free it can be expressed as the gradient of a scalar potential)

$$\vec{\nabla} \cdot \vec{a} = 0 \Rightarrow \vec{a} = \vec{\nabla} \times \vec{A} \tag{EA.8}$$

(In words: When a vector field is source free it can be expressed as the curl of a vector potential)

One identity between different operator expressions of a vector field which was used above is the following:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{a} = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \Delta\vec{a} \tag{EA.9}$$

A lot of other identities between the above operators in reference to scalar and vector fields can be found in literature. Also refer to literature in regard to the evaluation of surface and volume integrals as well as to relations between them like the Stokes, Gauss or Greens relations.